Sydney Grammar School

2009 Trial Examination

FORM VI
MATHEMATICS EXTENSION 2

Tuesday 11th August 2009

General Instructions
- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper
- Total marks — 120
- All eight questions may be attempted.
- All eight questions are of equal value.

Collection
- Write your candidate number clearly on each booklet and on the tear-off sheet.
- Hand in the eight questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Bundle the tear-off sheet with the question it belongs to.
- Bundle the tear-off sheet with the question it belongs to.
- Place the question paper inside your answer booklet for Question 1.

Checklist
- SGS booklets — 8 per boy
- Candidature — 72 boys

Examiner
DNW

QUESTION ONE (15 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\displaystyle\int_0^1 xe^{x^2}\,dx$. **2**

(b) Complete the square to find $\displaystyle\int \frac{dx}{x^2-2x+5}$. **2**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} x\sin x\,dx$. **3**

(d) (i) Find values of a, b and c such that **3**

$$\frac{x^2+2x-4}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4}\,.$$

(ii) Hence evaluate $\displaystyle\int_0^1 \frac{x^2+2x-4}{(x+1)(x^2+4)}\,dx$. **3**

(e) Use the substitution $x = \frac{\pi}{2} - u$ to show that **2**

$$\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin 2x}\,dx = 0\,.$$

QUESTION TWO (15 marks) Use a separate writing booklet. **Marks**

(a) Let $z = 3 - 4i$ and $w = 2 + i$. Find, in the form $x + iy$:

(i) $z + iw$ **1**

(ii) $z\,\overline{w}$ **1**

(b) Let $\alpha = 1 - i$.

(i) Write α in modulus-argument form. **1**

(ii) Hence show that $\alpha^4 + 4 = 0$. **2**

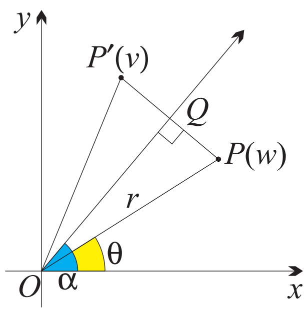
(c) Let $z = x + iy$ and $w = 1 - \dfrac{2i}{z}$.

(i) Write w in the form $a + ib$. **2**

(ii) For what value of z is w undefined? **1**

(iii) Given that w is purely imaginary, describe the locus of z. **2**

(d)



In the Argand diagram above, P represents the complex number $w = r\,\text{cis}\,\theta$. Q is that point on the ray $\arg(z) = \alpha$ such that $\angle PQO = \frac{\pi}{2}$. The point P', which represents the complex number v, is the reflection of P in the ray $\arg(z) = \alpha$. You may assume that $\triangle OPQ \equiv \triangle OP'Q$.

(i) Write down the values of $|v|$ and $\arg(v)$. **2**

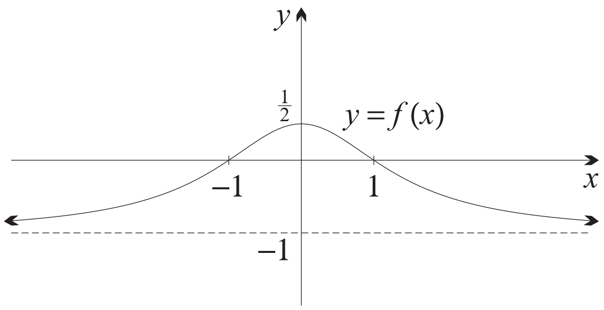
(ii) Hence show that $v = \overline{w}\,\text{cis}\,2\alpha$. **1**

(iii) The circle $|z - (2 + 2i)| = 1$ is reflected in the ray $\arg(z) = \frac{\pi}{6}$. By using the result in part (ii), or otherwise, show that the equation of this new circle is **2**

$$\left| z - \big((1 + \sqrt{3}) + i(\sqrt{3} - 1)\big) \right| = 1\,.$$

QUESTION THREE (15 marks) Use a separate writing booklet. **Marks**

(a)



The graph of $y = f(x)$ is shown above. The horizontal asymptote is $y = -1$ and the y-intercept is at $(0, \frac{1}{2})$. The x-intercepts are at $(-1, 0)$ and $(1, 0)$.

Draw separate graphs of the following functions:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = \Big(f(x)\Big)^2$ **2**

(iii) $y = 4^{f(x)}$ **2**

(b) The ellipse $\mathcal{E}$ has equation $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$.

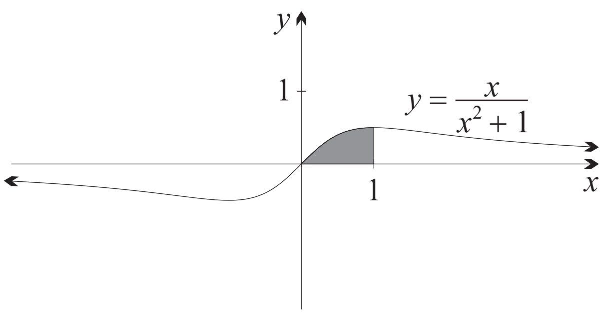
(i) State the intercepts with the axes. **1**

(ii) Determine the eccentricity of $\mathcal{E}$. **1**

(iii) State the coordinates of the two foci. **1**

(iv) Find the equations of the two directrices. **1**

(c) **5**



The graph of $y = \dfrac{x}{x^2 + 1}$ is shown above.

Use the method of cylindrical shells to find the volume of the solid generated when the shaded region bounded by $y = 0$, $y = \dfrac{x}{x^2 + 1}$ and $x = 1$ is rotated about the y-axis.

Exam continues next page ...

QUESTION FOUR (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Show that $2\cos A\,\sin B = \sin(A+B) - \sin(A-B)$. **1**

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{3}} 2\cos 2x\,\sin x\,dx$. **2**

(b) Consider the integral $I_n = \displaystyle\int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx$.

(i) Show that $I_0 = 2\sqrt{2} - 2$. **1**

(ii) Show that $I_{n-1} + I_n = \displaystyle\int_0^1 x^{n-1}\sqrt{1+x}\,dx$. **1**

(iii) Use integration by parts to show that **2**

$$I_n = \frac{2\sqrt{2} - 2nI_{n-1}}{2n+1}.$$

(iv) Hence evaluate I_2. **2**

(c) The number c is real and non-zero. It is also known that $(1+ic)^5$ is real.

(i) Use the binomial theorem to expand $(1+ic)^5$. **1**

(ii) Show that $c^4 - 10c^2 + 5 = 0$. **2**

(iii) Hence show that $c = \sqrt{5-2\sqrt{5}}$, $-\sqrt{5-2\sqrt{5}}$, $\sqrt{5+2\sqrt{5}}$ or $-\sqrt{5+2\sqrt{5}}$. **1**

(iv) Let $1 + ic = r\,\mathrm{cis}\,\theta$. Use de Moivre's theorem to show that the smallest positive value of θ is $\frac{\pi}{5}$. **1**

(v) Hence evaluate $\tan\frac{\pi}{5}$. **1**

Exam continues overleaf ...

QUESTION FIVE (15 marks) Use a separate writing booklet. **Marks**

(a) The polynomial $P(z) = 2z^3 - 3z^2 + 8z + 5$ has a zero at $z = 1 - 2i$. Factorise $P(z)$. **3**

(b) (i) The cubic equation $x^3 - px - q = 0$ has a double root. Show that $27q^2 = 4p^3$. **3**

(ii) Hence find the y-coordinates of the stationary points of $y = x^3 - 3x$ without the use of calculus. **1**

(c) Consider the series:

$$S = 1 - x^2 + x^4 - x^6 + \cdots .$$

(i) For which values of x does S have a limiting sum, and what is the limiting sum? **2**

(ii) Assuming that it is valid to integrate this series, show that **3**

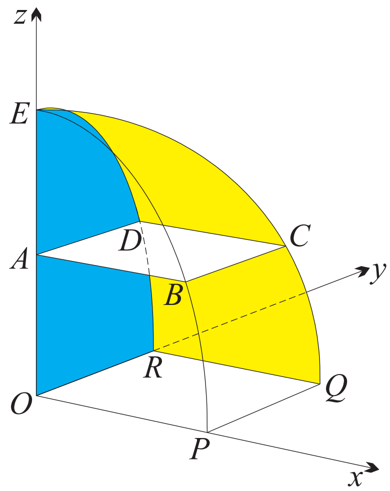
$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots .$$

(iii) Show that $\tan \frac{\pi}{12} = 2 - \sqrt{3}$. **2**

(iv) Let $x = \tan \frac{\pi}{12}$. Use this value of x and the first three terms of the series in part (ii) to find an approximation for π, correct to four decimal places. **1**

QUESTION SIX (15 marks) Use a separate writing booklet. **Marks**

(a)



The solid in the diagram above has a horizontal square base $OPQR$ with diagonal $OQ = r$. The thin horizontal slice $ABCD$ at height z above the base is also square with $OC = r$. The line OA is vertical. The curve QCE is a quadrant of a circle with centre O and radius r.

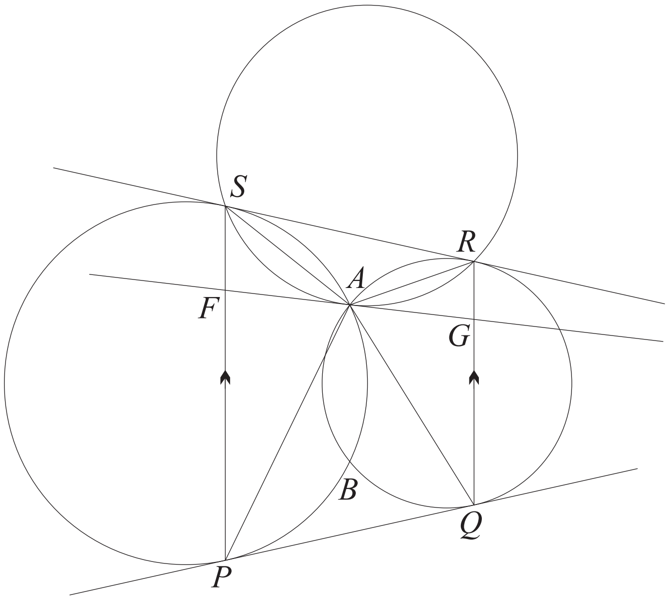
(i) Show that the area of $ABCD$ is $\frac{1}{2}(r^2 - z^2)$. **2**

(ii) Hence find the volume of the solid. **3**

(b) The points $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$ lie on the same branch of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and PQ is a focal chord, passing through $S(ae, 0)$. **4**

Use the gradients of PS and QS to show that $e = \dfrac{\sin\theta - \sin\phi}{\sin(\theta - \phi)}$.

(c)



In the diagram above, two circles of differing radius intersect at A and B. The lines PQ and RS are the common tangents with $PS||QR$. A third circle passes through the points S, A and R. The tangent to this circle at A meets the parallel lines at F and G.

Let $\angle RAG = \alpha$, $\angle AGR = \beta$ and $\angle GRA = \gamma$.

NOTE: You do not have to copy the diagram above. It has been reproduced for you on a tear-off sheet at the end of this paper. Insert this sheet into your answer booklet.

(i) State why $\angle AFP = \beta$. **1**

(ii) Show that $\angle SPA = \alpha$. **2**

(iii) Hence prove that FG is also tangent to the circle which passes through the points A, P and Q. **3**

Exam continues next page ...

QUESTION SEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) The definition of ${}^k\mathrm{C}_r$ is the coefficient of x^r in the expansion of $(1+x)^k$. Using this definition, what is the value of ${}^k\mathrm{C}_r$ whenever $k < r$? **1**

(ii) Prove that $\displaystyle\sum_{k=0}^{n} {}^k\mathrm{C}_r = {}^{n+1}\mathrm{C}_{r+1}$. You may assume the addition property for the binomial coefficients, which may be written as ${}^k\mathrm{C}_r = {}^{k+1}\mathrm{C}_{r+1} - {}^k\mathrm{C}_{r+1}$. **2**

(iii) Use the result proven in part (ii) to show that $\displaystyle\sum_{k=0}^{n} k = \tfrac{1}{2}n(n+1)$. **1**

(iv) (α) Show that $k^2 = 2 \times {}^k\mathrm{C}_2 + {}^k\mathrm{C}_1$. **1**

(β) Hence find a formula for $\displaystyle\sum_{k=0}^{n} k^2$. **2**

(b) Show that the equation of the directrix of the parabola $y = ax^2 + bx$ is **2**

$$y = -\frac{b^2+1}{4a}.$$

(c) A projectile is fired from the origin O with initial speed V and angle of projection α. The Cartesian equation of its trajectory is

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V^2}.$$

(i) Use part (b) to find the equation of the directrix. **2**

(ii) Hence show that the focus lies on the circle **1**

$$x^2 + y^2 = \frac{V^4}{4g^2}.$$

(iii) There is only one trajectory which passes through P. Use the geometry of the parabola to prove that OP is a focal chord. **3**

QUESTION EIGHT (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the function

$$f(x) = x - \frac{g^2}{x} - 2g\log\left(\frac{x}{g}\right), \quad \text{for } x \geq g\,.$$

(i) Evaluate $f(g)$. **1**

(ii) Show that $f'(x) = \left(1 - \frac{g}{x}\right)^2$. **1**

(iii) Explain why $f(x) > 0$ for $x > g$. **1**

(b) A body is moving vertically through a resisting medium, with resistance proportional to its speed. The body is initially fired upwards from the origin with speed v_0. Let y metres be the height of the object above the origin at time t seconds, and let g be the constant acceleration due to gravity. Thus

$$\frac{d^2y}{dt^2} = -(g + kv) \quad \text{where} \quad k > 0\,.$$

(i) Find v as a function of t, and hence show that **4**

$$k^2 y = (g + kv_0)(1 - e^{-kt}) - gkt\,.$$

(ii) Find T, the time taken to reach the maximum height. **1**

(iii) Show that when $t = 2T$, **1**

$$k^2 y = (g + kv_0) - \frac{g^2}{g + kv_0} - 2g\log\left(\frac{g + kv_0}{g}\right).$$

(iv) Use this result and part (a) to show that the downwards journey takes longer. **1**

(c) Suppose that the equation $f(x) = 0$ has a single root $x = \alpha$, where $a \leq \alpha \leq b$. Let the sequence

$$x_0 = a,\ x_1 = b,\ x_2 = \frac{a+b}{2},\ x_3,\ x_4,\ \ldots$$

be the successive approximations of $x = \alpha$ obtained when the bisection method is used. (The bisection method is also known as the method of halving the interval.)

Let $u_n = |x_n - x_{n-1}|$ be the distances between successive terms of this sequence.

(i) Explain why $u_{n+1} = \frac{1}{2}u_n$. **1**

(ii) Hence show that $u_n = (b - a)\left(\frac{1}{2}\right)^{n-1}$ for $n \geq 1$. **2**

(iii) Explain why $|\alpha - x_n| \leq u_n$. **1**

(iv) Hence prove that the bisection method converges to the root $x = \alpha$. **1**
That is, prove that $\lim_{n\to\infty} x_n = \alpha$.

END OF EXAMINATION

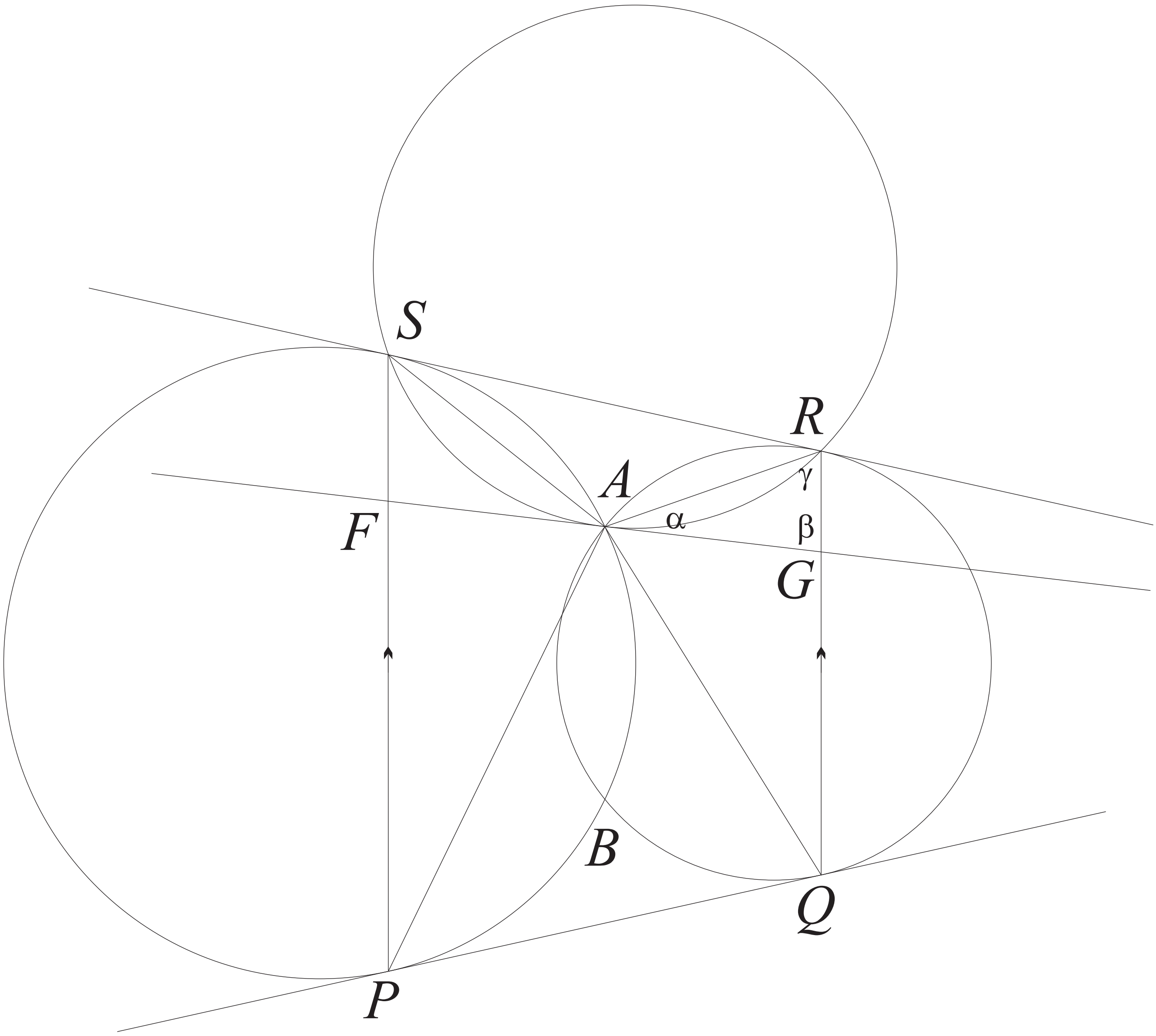
Tear-off pages follow ...

CANDIDATE NUMBER: ....

DETACH THIS SHEET AND BUNDLE IT WITH THE REST OF QUESTION SIX.

QUESTION SIX

(c)



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The following list of standard integrals may be used:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \;\; n \neq -1; \;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \;\; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \;\; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \;\; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \;\; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \;\; a > 0, \;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

$$\text{NOTE}: \;\; \ln x = \log_e x, \;\; x > 0$$

QUESTION ONE (15 marks)

(a) $$\int_0^1 xe^{x^2}\,dx = \tfrac{1}{2}\left[e^{x^2}\right]_0^1 \quad ✓$$
$$= \tfrac{1}{2}(e-1)\,. \quad ✓$$

(b) $$\int \frac{dx}{x^2-2x+5} = \int \frac{dx}{(x-1)^2+2^2} \quad ✓$$
$$= \tfrac{1}{2}\tan^{-1}\frac{x-1}{2} + C \quad ✓$$

(c) $$\int_0^{\frac{\pi}{2}} x\sin x\,dx = \Big[-x\cos x\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\,dx \quad \text{(by parts)} \quad ✓$$
$$= \Big[-x\cos x + \sin x\Big]_0^{\frac{\pi}{2}} \quad ✓$$
$$= 1 \quad ✓$$

(d) (i) The given equation is true if
$$x^2+2x-4 = a(x^2+4)+(bx+c)(x+1)\,.$$
At $x=-1$ $\quad -5 = 5a+0$
so $\quad a=-1\,.$ ✓
At $x=0$ $\quad -4=-4+c$
so $\quad c=0\,.$ ✓
At $x=1$ $\quad -1=-5+2b$
so $\quad b=2\,.$ ✓

(ii) $$\int_0^1 \frac{x^2+2x-4}{(x+1)(x^2+4)}\,dx = \int_0^1 \frac{2x}{x^2+4} - \frac{1}{x+1}\,dx \quad \text{(from part (i))} \quad ✓$$
$$= \Big[\log(x^2+4) - \log(x+1)\Big]_0^1 \quad ✓$$
$$= \log 5 - \log 2 - \log 4 + \log 1$$
$$= \log\left(\tfrac{5}{8}\right) \quad ✓$$

(e) Let $$I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin 2x}\,dx$$
Put $x = \frac{\pi}{2} - u$
then $dx = -du\,.$
at $x=0\,,\quad u=\frac{\pi}{2}$
and at $x=\frac{\pi}{2}\,,\quad u=0.$

Thus $\quad I = \displaystyle\int_{\frac{\pi}{2}}^{0} \frac{\cos(\frac{\pi}{2}-u)-\sin(\frac{\pi}{2}-u)}{1+\sin 2(\frac{\pi}{2}-u)}\,(-du)$ √

$$= \int_{0}^{\frac{\pi}{2}} \frac{\cos(\frac{\pi}{2}-u)-\sin(\frac{\pi}{2}-u)}{1+\sin 2(\frac{\pi}{2}-u)}\,du$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{\sin u-\cos u}{1+\sin 2u}\,du$$

so $\quad I = -I$ √

Hence $\quad I = 0$

Total for Question 1: 15 Marks

QUESTION TWO (15 marks)

(a) (i)
$$\begin{aligned} z + iw &= 3 - 4i + 2i - 1 \\ &= 2 - 2i \end{aligned}$$ √

(ii)
$$\begin{aligned} z\,\overline{w} &= (3-4i)(2-i) \\ &= 2 - 11i \end{aligned}$$ √

(b) (i) $\quad 1 - i = \sqrt{2}\,\mathrm{cis}(-\frac{\pi}{4})$ √

(ii)
$$\begin{aligned} \alpha^4 + 4 &= \left(\sqrt{2}\,\mathrm{cis}(-\tfrac{\pi}{4})\right)^4 + 4 \\ &= 4\,\mathrm{cis}(-\pi) + 4 \quad \text{(by de Moivre)} \\ &= -4 + 4 \\ &= 0 \end{aligned}$$ √ √

(c) (i)
$$\begin{aligned} w &= 1 - \frac{2i}{z} \\ &= 1 - \frac{2i\,\overline{z}}{|z|^2} \\ &= \left(1 - \frac{2y}{x^2+y^2}\right) - \frac{2ix}{x^2+y^2} \end{aligned}$$ √ √

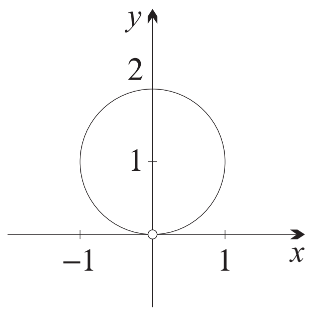
(ii) w is undefined when $z = 0$ √

(iii) Since w is pure imaginary,

$$\mathrm{Re}(w) = 0$$

so $\quad x^2 + y^2 - 2y = 0$ √

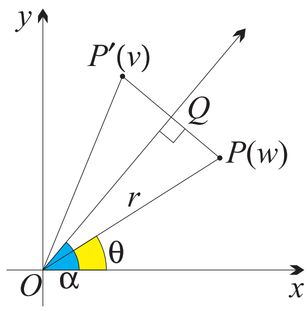
or $\quad x^2 + (y-1)^2 = 1$



Thus the locus is the unit circle with centre $z = i$, omitting the origin. √

(d) (i)

$$|v| = |w| \quad (\text{since } OP' = OP)$$
$$= r$$
$$\arg v = \alpha + \angle P'OQ$$
$$= \alpha + \angle POQ$$
$$= \alpha + (\alpha - \theta)$$
$$= 2\alpha - \theta$$



√ √

(ii)

$$v = r\,\text{cis}(2\alpha - \theta)$$
$$= r\,\text{cis}(-\theta)\,\text{cis}\,2\alpha$$
$$= \overline{w}\,\text{cis}\,2\alpha$$

√

(iii) The radius remains the same for a reflection. The new centre will be

$$\overline{(2+2i)}\,\text{cis}(2 \times \tfrac{\pi}{6}) = (2-2i)\,\text{cis}\,\tfrac{\pi}{3}$$ √
$$= (2-2i)\tfrac{1}{2}(1+i\sqrt{3})$$
$$= (1+\sqrt{3}) + i(\sqrt{3}-1)$$ √

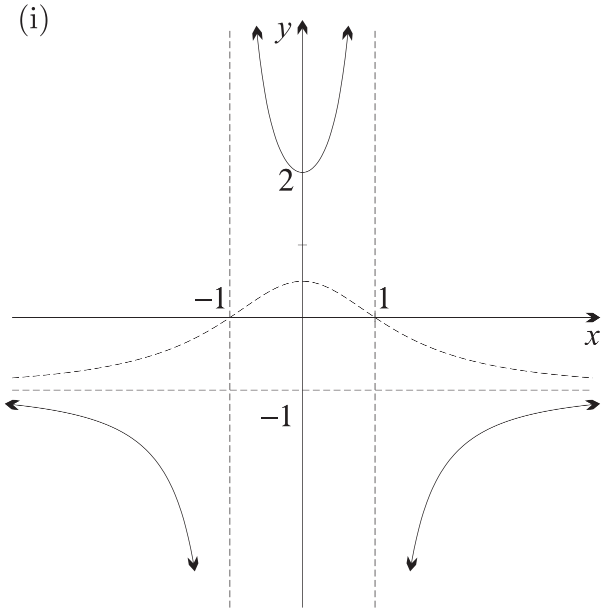
Hence the new circle is $\left|z - \big((1+\sqrt{3}) + i(\sqrt{3}-1)\big)\right| = 1$.

Total for Question 2: 15 Marks

QUESTION THREE (15 marks)

(a) The graphs below are exact.

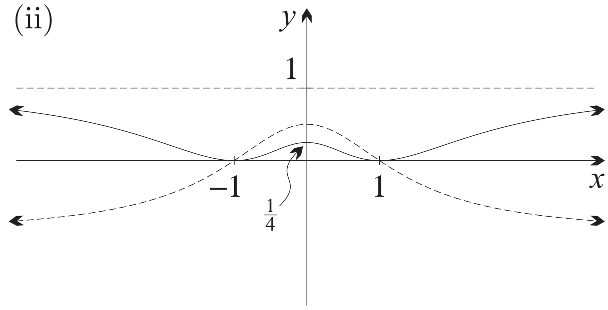
(i)



vertical asymptotes √

y-intercept and horizontal asymptote √

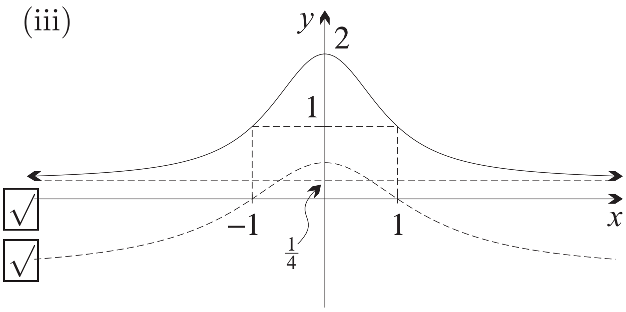
(ii)



Shape at x-intercepts √

y-intercept and horizontal asymptote √

(iii)



$(-1, 1)$ and $(1, 1)$ √

y-intercept and horizontal asymptote √

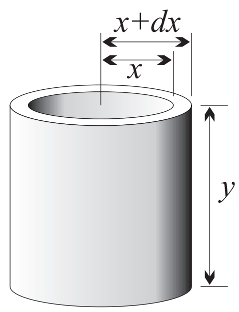
(b) (i) $(5,0), (-5,0), (0,4)$ and $(0,-4)$ ✓

(ii) From
$$b^2 = a^2(1-e^2)$$
$$16 = 25(1-e^2)$$
$$e^2 = \tfrac{9}{25}$$
so
$$e = \tfrac{3}{5}$$ ✓

(iii) $(3,0)$ and $(-3,0)$ ✓

(iv) $x = \frac{25}{3}$ and $x = -\frac{25}{3}$ ✓

(c) The volume of the element is the difference between two cylinders, thus



$$dV = \pi(x+dx)^2 y - \pi x^2 y$$
$$= \pi y(2x+dx)dx$$

Sum the elements and take the limit as $dx \to 0$ to get

$$V = \int_0^1 2\pi xy\,dx$$ ✓
$$= 2\pi \int_0^1 \frac{x^2}{x^2+1}\,dx$$ ✓
$$= 2\pi \int_0^1 1 - \frac{1}{x^2+1}\,dx$$ ✓
$$= 2\pi \Big[x - \tan^{-1} x\Big]_0^1$$ ✓
$$= 2\pi - \tfrac{\pi^2}{2}$$ ✓

Total for Question 3: 15 Marks

QUESTION FOUR (15 marks)

(a) (i)
$$RHS = \quad \sin A\cos B + \cos A \sin B$$
$$\underline{-\sin A \cos B + \cos A \sin B}$$ ✓
$$= 2\cos A \sin B$$

(ii)
$$\int_0^{\frac{\pi}{3}} 2\cos 2x\,\sin x\,dx = \int_0^{\frac{\pi}{3}} \sin 3x - \sin x\,dx$$ ✓
$$= \Big[-\tfrac{1}{3}\cos 3x + \cos x\Big]_0^{\frac{\pi}{3}}$$
$$= (\tfrac{1}{3} + \tfrac{1}{2}) - (-\tfrac{1}{3} + 1)$$
$$= \tfrac{1}{6}$$ ✓

(b) (i)

$$I_0 = \int_0^1 \frac{1}{\sqrt{1+x}}\,dx$$
$$= 2\Big[\sqrt{1+x}\Big]_0^1 \quad \checkmark$$
$$= 2\sqrt{2} - 2\,.$$

(ii)

$$LHS = \int_0^1 \frac{x^{n-1}}{\sqrt{1+x}} + \frac{x^n}{\sqrt{1+x}}\,dx$$
$$= \int_0^1 \frac{x^{n-1}(1+x)}{\sqrt{1+x}}\,dx \quad \checkmark$$
$$= \int_0^1 x^{n-1}\sqrt{1+x}\,dx$$
$$= RHS\,.$$

(iii)

$$I_n = \Big[2x^n\sqrt{1+x}\Big]_0^1 - 2n\int_0^1 x^{n-1}\sqrt{1+x}\,dx \quad \text{(by parts)} \quad \checkmark$$
$$= 2\sqrt{2} - 2n\,(I_{n-1} + I_n) \quad \text{(by part ii)}$$

so $(2n+1)I_n = 2\sqrt{2} - 2nI_{n-1}$ ✓

or $$I_n = \frac{2\sqrt{2} - 2nI_{n-1}}{2n+1}\,.$$

(iv)

$$I_1 = \tfrac{1}{3}\left(2\sqrt{2} - 2I_0\right)$$
$$= \tfrac{1}{3}(4 - 2\sqrt{2})\,. \quad \checkmark$$
$$I_2 = \tfrac{1}{5}\left(2\sqrt{2} - 4I_1\right)$$
$$= \tfrac{1}{5}\left(2\sqrt{2} - \tfrac{16}{3} + \tfrac{8\sqrt{2}}{3}\right)$$
$$= \tfrac{1}{15}(14\sqrt{2} - 16)\,. \quad \checkmark$$

(c) (i)

$$(1+ic)^5 = 1 + 5ic - 10c^2 - 10ic^3 + 5c^4 + ic^5 \quad \checkmark$$

(ii)

$$\text{Im}\left((1+ic)^5\right) = 0 \quad \checkmark$$

so $5c - 10c^3 + c^5 = 0$

thus $c^4 - 10c^2 + 5 = 0$ (since $c \neq 0$) ✓

(iii) The equation is a quadratic in c^2, thus

$$c^2 = \frac{10+\sqrt{80}}{2} \text{ or } \frac{10-\sqrt{80}}{2} \quad \checkmark$$

hence $c = \sqrt{5-2\sqrt{5}}\,, -\sqrt{5-2\sqrt{5}}\,, \sqrt{5+2\sqrt{5}}$ or $-\sqrt{5+2\sqrt{5}}\,.$

(iv)

$$(r\,\text{cis}\,\theta)^5 = r^5\,\text{cis}\,5\theta \quad \text{(by de Moivre)}$$

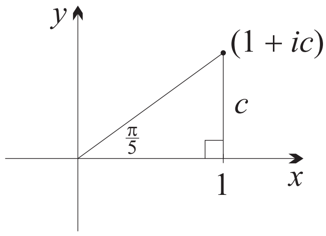
and since this is real

$$\sin 5\theta = 0$$
$$5\theta = n\pi$$

or $\theta = \frac{n\pi}{5}$ ✓

Thus the smallest positive value is $\theta = \frac{\pi}{5}\,.$

(v) This corresponds to the smallest positive value of c.



Thus $\tan\frac{\pi}{5} = \dfrac{c}{1}$

$= \sqrt{5 - 2\sqrt{5}}\,.$ √

Total for Question 4: 15 Marks

QUESTION FIVE (15 marks)

(a) Since $P(z)$ has real coefficients, it follows that $z = 1 + 2i$ is also a zero. √
Let the remaining zero be α, then summing the roots

$\alpha + 2 = \frac{3}{2}$

or $\alpha = -\frac{1}{2}$ √

Hence $P(z) = 2\left(z + \frac{1}{2}\right)\big(z - (1 - 2i)\big)\big(z - (1 + 2i)\big)\,.$ √

(b) (i) Let the roots be α, α and β, then by the sums and products of roots

$$2\alpha + \beta = 0 \qquad (1)$$
$$\alpha^2 + 2\alpha\beta = -p \qquad (2)$$
$$\alpha^2\beta = q \qquad (3)$$ √

From (1), equations (2) and (3) become

$$3\alpha^2 = p \qquad (4)$$
$$2\alpha^3 = -q \qquad (5)$$ √

hence $4p^3 = 4 \times 27\alpha^6$ (from equation (4))

$= 27 \times 4\alpha^6$ √

$= 27q^2$ (from equation (5).)

(ii) Re-writing the equation of the cubic

$x^3 - 3x - y = 0$

which has a double root at the y-coordinates of the stationary points, so

$27y^2 = 4 \times 3^3$ (from part (i))

so $y^2 = 4$

thus $y = 2$ or -2 √

(c) (i) $|x| < 1$

or $-1 < x < 1$ √

for which $S = \dfrac{1}{1 + x^2}$ √

(ii) Thus $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$

so $\displaystyle\int \frac{dx}{1+x^2} = \int \left(1 - x^2 + x^4 - x^6 + \cdots\right) dx$

and $\tan^{-1} x = \left(x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots\right) + C$ ✓✓

At $x = 0$, $\tan^{-1} 0 = 0$, so

$$C = 0$$ ✓

thus $\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots$.

(iii)
$$\begin{aligned} \tan \tfrac{\pi}{12} &= \tan\left(\tfrac{\pi}{3} - \tfrac{\pi}{4}\right) \quad ✓ \\ &= \frac{\sqrt{3}-1}{1+\sqrt{3}} \\ &= \frac{(\sqrt{3}-1)^2}{2} \quad ✓ \\ &= 2 - \sqrt{3}\,. \end{aligned}$$

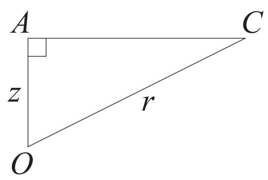
(iv)
$$\tfrac{\pi}{12} \doteqdot (2-\sqrt{3}) - \tfrac{1}{3}(2-\sqrt{3})^3 + \tfrac{1}{5}(2-\sqrt{3})^5$$

so
$$\begin{aligned} \pi &\doteqdot 12\Big((2-\sqrt{3}) - \tfrac{1}{3}(2-\sqrt{3})^3 + \tfrac{1}{5}(2-\sqrt{3})^5\Big) \\ &\doteqdot 3{\cdot}1418 \quad \text{(correct to four decimal places)} \quad ✓ \end{aligned}$$

Total for Question 5: 15 Marks

<u>QUESTION SIX</u> (15 marks)

(a) (i)



In $\triangle OAC$ $AC^2 = r^2 - z^2$ (by Pythagoras) ✓

hence $|ABCD| = \tfrac{1}{2}AC^2$ (a square is a rhombus) ✓

$= \tfrac{1}{2}(r^2 - z^2)$.

(ii) The volume of the thin slice with thickness dz is $dV = \tfrac{1}{2}(r^2 - z^2)\,dz$

Sum the elements and take the limit as $dz \to 0$ to get

$$\begin{aligned} V &= \tfrac{1}{2}\int_0^r (r^2 - z^2)\,dz \quad ✓ \\ &= \tfrac{1}{2}\left[r^2 z - \tfrac{1}{3}z^3\right]_0^r \quad ✓ \\ &= \tfrac{1}{2}(r^3 - \tfrac{1}{3}r^3) - 0 \\ &= \tfrac{1}{3}r^3\,. \quad ✓ \end{aligned}$$

(b) Since S lies on PQ it follows that

$$\text{gradient } PS = \text{gradient } QS \qquad \surd$$

thus
$$\frac{b\tan\theta}{a\sec\theta - ae} = \frac{b\tan\phi}{a\sec\phi - ae}$$

or
$$\frac{\tan\theta}{\sec\theta - e} = \frac{\tan\phi}{\sec\phi - e} \qquad \surd$$

whence
$$\tan\theta\sec\phi - e\tan\theta = \tan\phi\sec\theta - e\tan\phi$$

and
$$e(\tan\theta - \tan\phi) = \tan\theta\sec\phi - \tan\phi\sec\theta\,. \qquad \surd$$

So
$$\begin{aligned} e &= \frac{\tan\theta\sec\phi - \tan\phi\sec\theta}{\tan\theta - \tan\phi} \times \frac{\cos\theta\cos\phi}{\cos\theta\cos\phi} \\ &= \frac{\sin\theta - \sin\phi}{\sin\theta\cos\phi - \cos\theta\sin\phi} \qquad \surd \\ &= \frac{\sin\theta - \sin\phi}{\sin(\theta - \phi)}\,. \end{aligned}$$

(c) (i) $\angle AFP = \beta$ (Alternate angles, $PS||QR$.) $\surd$

(ii) $\angle RSA = \angle RAG$ (angle in the alternate segment of circle SAR) $\surd$

$= \alpha\,.$

$\angle SPA = \angle RSA$ (angle in the alternate segment of circle $PBAS$) $\surd$

$= \alpha\,.$

(iii) $\angle FAP = \gamma$ (angle sum of $\triangle FAP$) $\surd$

$\angle PQA = \angle QRA$ (angle in the alternate segment of circle $RABQ$) $\surd$

$= \gamma\,.$

Thus $\angle FAP = \angle PQA$

Hence FG is tangent to the circle through APQ by the converse of the angles in the alternate segment theorem. $\surd$

Total for Question 6: 15 Marks

QUESTION SEVEN (15 marks)

(a) (i) If $k < r$ then there is no term in x^r, hence ${}^k\mathrm{C}_r = 0$. ☑

(ii)
$$\begin{aligned}\sum_{k=0}^{n} {}^k\mathrm{C}_r &= \sum_{k=0}^{n} \left({}^{k+1}\mathrm{C}_{r+1} - {}^k\mathrm{C}_{r+1}\right)\\ &= \left({}^1\mathrm{C}_{r+1} - {}^0\mathrm{C}_{r+1}\right) + \left({}^2\mathrm{C}_{r+1} - {}^1\mathrm{C}_{r+1}\right) + \left({}^3\mathrm{C}_{r+1} - {}^2\mathrm{C}_{r+1}\right)\\ &\quad + \ldots + \left({}^{n+1}\mathrm{C}_{r+1} - {}^n\mathrm{C}_{r+1}\right)\\ &= {}^{n+1}\mathrm{C}_{r+1} - {}^0\mathrm{C}_{r+1} \quad \text{(since all other terms cancel)}\\ &= {}^{n+1}\mathrm{C}_{r+1} - 0 \quad \text{(by part (i))}\\ &= {}^{n+1}\mathrm{C}_{r+1}\end{aligned}$$
☑ ☑

(iii)
$$\begin{aligned}\sum_{k=0}^{n} k &= \sum_{k=0}^{n} {}^k\mathrm{C}_1\\ &= {}^{n+1}\mathrm{C}_2 \quad \text{(by part (ii))}\\ &= \tfrac{1}{2}n(n+1)\,.\end{aligned}$$
☑

(iv) (α)
$$\begin{aligned}2 \times {}^k\mathrm{C}_2 + {}^k\mathrm{C}_1 &= k(k-1) + k\\ &= k^2\,.\end{aligned}$$
☑

(β)
$$\begin{aligned}\sum_{k=0}^{n} k^2 &= \sum_{k=0}^{n} 2 \times {}^k\mathrm{C}_2 + {}^k\mathrm{C}_1\\ &= 2 \times {}^{n+1}\mathrm{C}_3 + {}^{n+1}\mathrm{C}_2 \quad \text{(by part (ii))}\\ &= \tfrac{1}{3}(n+1)n(n-1) + \tfrac{1}{2}(n+1)n\\ &= \tfrac{1}{6}(n+1)n\big(2(n-1)+3\big)\\ &= \tfrac{1}{6}(n+1)n(2n+1)\,.\end{aligned}$$
☑ ☑

(b) The focal length $= \frac{1}{4a}$ ☑

At the vertex
$$\begin{aligned}y &= x(ax+b)\\ &= -\tfrac{b}{2a}\left(-\tfrac{b}{2} + b\right)\\ &= -\tfrac{b^2}{4a}\end{aligned}$$

hence the directrix has equation
$$\begin{aligned}y &= -\tfrac{b^2}{4a} - \tfrac{1}{4a}\\ &= -\tfrac{b^2+1}{4a}\,.\end{aligned}$$
☑

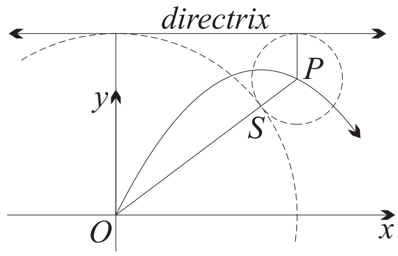
(c) (i)
$$\begin{aligned}y &= -\frac{\tan^2\alpha + 1}{4\left(\frac{-g\sec^2\alpha}{2V^2}\right)}\\ &= \frac{V^2\sec^2\alpha}{2g\sec^2\alpha}\\ &= \frac{V^2}{2g}\end{aligned}$$
☑ ☑

(ii) The origin lies on the parabola so is equidistant from the focus and directrix. Thus there is a circle with centre the origin which passes through the focus and is tangent to the directrix.

Hence the radius of this circle is $\dfrac{V^2}{2g}$ ☑

and the equation is $x^2 + y^2 = \dfrac{V^4}{4g^2}$.

(iii) Since P is on the parabola, P is equidistant from the focus and directrix. Hence there is a second circle with centre P which passes through the focus and is tangent to the directrix. ☑



Since there is only one trajectory, the two circles intersect once only, at S. ☑

Whence the two circles have a common tangent at S, which is perpendicular to both radii OS and SP. Hence O, S and P are collinear. ☑

That is, OP is a focal chord.

Total for Question 7: 15 Marks

QUESTION EIGHT (15 marks)

(a) (i)
$$\begin{aligned} f(g) &= g - \frac{g^2}{g} - 2g\log\left(\frac{g}{g}\right) \\ &= g - g - 2g\log 1 \\ &= 0 \end{aligned}$$ ☑

(ii)
$$\begin{aligned} f'(x) &= 1 + \frac{g^2}{x^2} - \frac{2g}{x} \\ &= \left(1 - \frac{g}{x}\right)^2 . \end{aligned}$$ ☑

(iii) Now $f(g) = 0$

and $f'(x) > 0$ for $x \neq g$ (that is, $f(x)$ increasing for $x > g$) ☑

hence $f(x) > 0$ for $x > g$.

(b) (i)
$$v' = -(g + kv)$$
so
$$\frac{kv'}{g + kv} = -k$$
Integrate with respect to t to get
$$\int \frac{kv'}{g + kv}\,dt = \int -k\,dt\,. \qquad \boxed{\surd}$$
Note on the LHS that the numerator is the derivative of the denominator

so $\quad \log(g + kv) = -kt + C_1 \quad$ (for some constant C_1)

or $\quad g + kv = Ae^{-kt} \quad$ where $\quad A = e^{C_1}\,.$

At $t = 0 \quad g + kv_0 = A$

so $\quad g + kv = (g + kv_0)e^{-kt}\,. \qquad \boxed{\surd}$

Integrate again with respect to t to get
$$gt + ky = -\tfrac{1}{k}(g + kv_0)e^{-kt} + \tfrac{1}{k}C_2 \quad \text{(for some constant } C_2\text{)}$$
so $\quad kgt + k^2y = -(g + kv_0)e^{-kt} + C_2\,. \qquad \boxed{\surd}$

At $t = 0 \quad 0 + 0 = -(g + kv_0) + C_2$

so $\quad C_2 = (g + kv_0)\,. \qquad \boxed{\surd}$

Thus $\quad kgt + k^2y = (g + kv_0)\left(1 - e^{-kt}\right)$

or $\quad ky^2 = (g + kv_0)\left(1 - e^{-kt}\right) - gkt\,.$

(ii) At $t = T$, $v = 0$ so
$$g = (g + kv_0)e^{-kT}$$
or $\quad e^{kT} = \dfrac{g + kv_0}{g}$

so $\quad T = \frac{1}{k}\log\left(\dfrac{g + kv_0}{g}\right)\,. \qquad \boxed{\surd}$

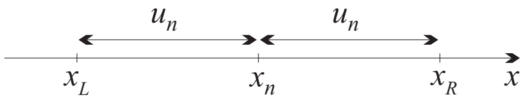
(iii) At $t = 2T$,
$$\begin{aligned} k^2y &= (g + kv_0)\left(1 - e^{-2kT}\right) - 2gkT \\ &= (g + kv_0)\left(1 - \left(e^{-kT}\right)^2\right) - 2gkT \\ &= (g + kv_0)\left(1 - \left(\frac{g}{g + kv_0}\right)^2\right) - 2g\log\left(\frac{g + kv_0}{g}\right) \qquad \boxed{\surd} \\ &= (g + kv_0) - \frac{g^2}{g + kv_0} - 2g\log\left(\frac{g + kv_0}{g}\right)\,. \end{aligned}$$

(iv) Let $x = g + kv_0$ then at $t = 2T$
$$\begin{aligned} k^2y &= x - \frac{g^2}{x} - 2g\log\left(\frac{x}{g}\right) \\ &= f(x) \\ &> 0 \quad \text{(by part (a))} \qquad \boxed{\surd} \end{aligned}$$
That is, it is above the ground and hence the downwards journey takes longer.

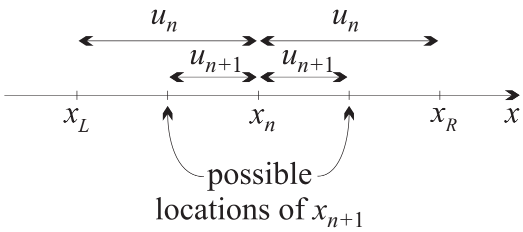
(c) (i) The value x_n is the average of two numbers, one of which is x_{n-1}. Let these two numbers be x_L and x_R, where $x_L < x_n < x_R$. Thus

$$u_n = x_n - x_L = x_R - x_n \, .$$

The situation is shown on the number line.



Either x_{n+1} is the mid-point of x_L and x_n
or x_{n+1} is the mid-point of x_n and x_R, as shown in the diagram.



In either case the $|x_{n+1} - x_n| = \frac{1}{2}|x_n - x_{n-1}|$, viz $u_{n+1} = \frac{1}{2}u_n$. $\boxed{\surd}$

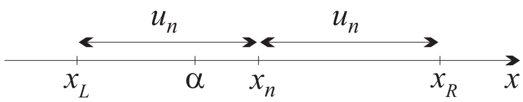
(ii) Since $\dfrac{u_{n+1}}{u_n} = \dfrac{1}{2}$ for all n, u_n is a GP with common ratio $= \dfrac{1}{2}$. $\boxed{\surd}$

First term $\quad u_1 = |b - a|$

$\qquad\qquad = (b - a)$ $\boxed{\surd}$

Hence $\quad u_n = (b - a)\left(\frac{1}{2}\right)^{n-1}$

(iii) The root α lies between x_L and x_R, as in the diagram below.



Hence the distance from α to x_n is less than or equal to u_n. $\boxed{\surd}$
That is $|\alpha - x_n| \leq u_n$.

(iv)
$$\begin{aligned} \lim_{n\to\infty} |\alpha - x_n| &\leq \lim_{n\to\infty} u_n \\ &\leq \lim_{n\to\infty} (b - a)\left(\tfrac{1}{2}\right)^{n-1} \\ &\leq 0 \end{aligned}$$

hence $\lim_{n\to\infty} |\alpha - x_n| = 0$ $\boxed{\surd}$

thus $\quad \lim_{n\to\infty} x_n = \alpha \, .$

Total for Question 8: 15 Marks